Chapter 6: Polarization and Crystal Optics

* P6-1. **Cascaded Wave Retarders.** Show that two cascaded quarter-wave retarders with parallel fast axes are equivalent to a half-wave retarder. What is the result if the fast axes are orthogonal.

P6-2. **Jones Matrix of a Polarizer.** Show that the Jones matrix of a linear polarizer with a transmission axis making an angle \( \theta \) with the \( x \) axis is

\[
T = \begin{bmatrix}
\cos^2 \theta & \sin \theta \cos \theta \\
\sin \theta \cos \theta & \sin^2 \theta
\end{bmatrix}
\]

(FoP 6.1-25)

Hint! Derive (FoP 6.1-25) using (FoP 6.1-18), (FoP 6.1-22) and (FoP 6.1-24)

* P6-3. Three polarizers are placed after each other. The first is illuminated by unpolarized light with the intensity \( I_0 \). The transmission direction for the second and the third polarizer is rotated 45° and 90° in relation to the first respectively. See figure 6-1.

![Figure 6-1](image)

*P6-4. Two polarizers can be used as a continuously variable grey filter. What is the angle between the transmission axes, so that 5.0 % of incoming light is transmitted? We assume that we can neglect reflections.

P6-5. Give the propagation direction and polarization state for the following waves:

a) \( \tilde{E} = \text{Re}\left(2i\hat{y} - 3\hat{z}\right) e^{(kz - \omega t)} \)
b) \( \tilde{E} = \text{Re}\left(\hat{x} - i\hat{z}\right) e^{(ky - \omega t)} \)
c) \( \tilde{E} = \text{Re}\left(\hat{x} + \hat{y} \cdot 3e^{i\pi/6}\right) e^{(kz - \omega t)} \)
* P6-6. Give the polarization state of the following wave:
  a) \( \vec{E} = \hat{x} \cdot E_0 \cdot \cos(kz - \omega t) + \hat{y} \cdot E_0 \cdot \cos(kz - \omega t - \pi / 2) \)
  b) \( \vec{E} = \hat{x} \cdot E_0 \cdot \cos(\omega t - kz) + \hat{y} \cdot E_0 \cdot \cos(\omega t - kz - \pi / 2) \)
  c) \( \vec{E} = \hat{x} \cdot E_0 \cdot \cos(\omega t - kz) + \hat{y} \cdot E_0 \cdot \cos(\omega t - kz - \pi) \)
  d) \( \vec{E} = \hat{x} \cdot E_0 \cdot \cos(\omega t - kz) - \hat{y} \cdot E_0 \cdot \cos(\omega t - kz - \pi / 4) \)
  e) \( \vec{E} = \text{Re} \{2i\hat{y} - 3\hat{z}\} \cdot e^{i(\omega t - kz)} \)

* P6-7. A plane, linearly polarized light wave, with intensity \( I_0 \), is transmitted through a system of perfect linear polarizers (we assume that all light is transmitted in the transmission direction but in the perpendicular direction all light is absorbed).
  Give for the following systems of polarizers and transmission directions the total transmitted intensity: (angles are measured in the same direction and relatively to the polarization direction of the incident light).
  a) one at 90° angle
  b) two at the angles 45° and 90°.
  c) three at the angles 30°, 60° and 90°.
  d) \( N \) polarizers an the angles 90°/2⋅90°/N, 3⋅90°/N, 4⋅90°/N,......90°
  e) from d) we let \( N \to \infty \)

P6-8. **Brewster window.** At what angle is a TM-polarized beam of light transmitted through a glass plate of refractive index \( n = 1.5 \) placed in air (\( n = 1 \)) without suffering reflection losses at either surface? Such plates, known as Brewster windows (figure 6-2), are used in lasers, as described in FoP Sec. 15.D.

![Figure 6-2](image)

* Figure 6-2. The Brewster window transmits TM-polarized light with no reflection loss.

* P6-9. **Reflectance of Glass.** A plane wave is incident from air (\( n = 1 \)) onto a glass plate (\( n = 1.5 \)) at an angle of incidence of 45°. Determine the power reflectances of the TE and TM waves. What is the average reflectance for unpolarized light (light carrying TE and TM waves of equal intensities)?

P6-10. Left elliptically polarized light impinges from air (\( n = 1 \)) to a glass surface (\( n = 1.56 \)). The ratio of the long and short axis of the ellipse is 3:2 and the long axis is in the plane of incidence. Calculate the angle of incidence for which the reflected light is right circularly polarized.
P6-11. Light impinges on a glass plate with the refractive index 1.5.
   a) Calculate the Brewster angle.
   b) Calculate the angle between the reflected beam and the transmitted beam.

* P6-12. When white light is reflected on a glass plate one can obtain plane polarized light if the plate is oriented at Brewster angle.
   a) Calculate the efficiency of the polarizer e.g. the intensity ratio between reflected plane polarized light and incident unpolarized light. The refractive index of the glass is 1.54.
   b) Higher efficiency can be obtained if transmitted light is used instead. In this case the light will not be perfectly polarized. It is then better to use several glass plates mounted in Brewster angle after each other. Calculate the number of plates needed so that the polarization degree* is higher than 99%. As in a) the refractive index of the glass is 1.54

* Polarisation degree is given by: \( P_{\text{deg}} = \frac{I_{\text{par}}}{I_{\text{par}} + I_{\text{perp}}} \)

P6-13. Start with Fresnel equations (FoP 6.2-8 and FoP 6.2-9) and show that when light impinges along the normal we have the following formula:

\[
\Re = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2
\]

Utilize that, for small angles, we can approximate \( \sin \theta \) and \( \tan \theta \) with the angle \( \theta \).

* P6-14.
   a) Calculate the thickness of the thinnest possible quarter wave plate of crystalline quartz with \( n_0 = 1.5497 \), \( n_e = 1.5590 \) for the vacuum wavelength 486 nm. Such a plate can be made thicker if it consists of two plates sandwiched together that counteract.
   b) A plane monochromatic, linearly polarized wave is incident on two thin quarter wave retarders according to a). They are placed after each other so that their optical axes have the angle \( \alpha \) respectively \( \beta \) relative to the incident linear polarization direction. Give for the following combinations of \( \alpha \) and \( \beta \) the final polarization state after the two retarders.

In the case when the resulted light is elliptically polarized, give the ratio between the short axis and the long axis in the ellipse describing the rotation of the \( E \)-vector. It is not necessary to give the rotation direction.

<table>
<thead>
<tr>
<th>Angles</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>45°</td>
<td>45°</td>
<td>0°</td>
<td>0°</td>
</tr>
<tr>
<td>( \beta )</td>
<td>45°</td>
<td>0°</td>
<td>45°</td>
<td>20°</td>
</tr>
</tbody>
</table>
P6-15. Between two crossed polarizers (figure 6-3) there is a glass plate with thickness $d$. This plate can be slightly birefringent if a mechanical force is applied at 45° relative to the two linear polarizers. Give how the transmitted irradiance $I$ after the second polarizer depends on the incident intensity $I_o$ on the plate, the thickness $d$, the vacuum wavelength $\lambda_0$ and $\Delta n = |n_e - n_o|$ = the difference between the extraordinary and the ordinary refractive index for the plate.

![Figure 6-3. Adjusting the transmitted beam by a birefringent plate.](image)

* P6-16. We have the following set-up (figure 6-5) for the wavelength 0.53 μm.

![Figure 6-4. A Faraday rotator combined with an optically active crystal.](image)

After a linear polarizer there is a Faraday rotator that turns the polarization direction 45°. This rotator is followed by an optically active crystal that turns the polarization direction back to the original direction. Answer the following questions:

a) A certain glass type with the Verdet constant $5.28^\circ (T \cdot cm)^{-1}$ is used in the Faraday rotator. It is possible to use a magnetic field of 1.0 T. How long glass rod has to be used?

b) As optically active crystal, a quartz crystal is used. The refractive index is $n_\parallel = 1.544204$ and $n_\perp = 1.544271$. How thick must the crystal be?

c) After the optically active crystal there is a mirror that reflects the light back the same path as it came from. Give the polarisation state after the optically active crystal and after the Faraday rotator. Give also the part (in %) of the reflex that passes back through the first linear polarizer.
* P6-17. A Wollaston prism according to the figure 6-5 is used. The surrounding medium is air and the optical axes are marked.

\[\begin{align*}
\text{Figure 6-5. A Wollaston prism.}
\end{align*}\]

The two parts of the prism are in contact, e.g. we can neglect any influence of air or glue at the border surface. The material is calcite with \(n_r = 1.6584\) and \(n_c = 1.4864\). Unpolarized light is impinging from left. Draw the rays through the material with an accurate mark of the polarization states. Calculate also the angle between the resulting rays.

* P6-18 A 0.900 mm thick plane parallel plate of crystalline quartz has the end surfaces parallel with the optical axes. It is placed between two polarizers which have their transmission directions parallel so that the optical axis of the quartz plate makes an angle of 45° with the transmission axes of the polarizers. A number of wavelengths are missing e.g. the two adjacent wavelengths 443 nm and 467 nm. Calculate \(\Delta n = |n_c - n_r|\).

P6-19. The following figure (figure 6-6) shows the cross section of a Rochon prism with the optical axes marked. The prism is of calcite with \(n_r = 1.6584\) and \(n_c = 1.4864\). Draw the rays through the prism for an incident ray of unpolarized light that impinges perpendicular to the surface on the left. Mark the polarization state of the beam and calculate also the angle between the two emergent beams.

\[\begin{align*}
\text{Figure 6-6. A Rochon prism.}
\end{align*}\]
Answers chapter 6:

P6-1: If orthogonal then no retardation is obtained.

P6-3: $I_0/2, I_0/4$ and $I_0/8$

P6-4: $72^\circ$

P6-5: a) in the positive x-direction, left elliptical polarisation
   b) in the positive y-direction, left circular polarisation
   c) in the positive z-direction, right elliptical polarization

P6-6: a) right circularly polarized b) left circularly polarized c) linearly polarized d) right elliptically polarized e) right elliptically polarized

P6-7: a) 0 b) 0.25$I_0$ c) 0.42$I_0$ d) $I_0 \left[ \cos \left( \frac{\pi}{2N} \right)^{2N} \right]$ e) $I_0$

P6-8: $56.3^\circ$

P6-9: TE: 0.092 TM: 0.0085 For unpolarized light 0.05

P6-10: $29.7^\circ$

P6-11: a) $56.3^\circ$ b) $90^\circ$ This value is independent of $n$

P6-12: a) 8.27 % b) at least 13 plates
P6-14: a) $13 \cdot 10^{-6}$ m

b) Optical axis

\[
\begin{array}{cccc}
I & 45^\circ & \text{Circular} & 45^\circ & \text{Linear} \\
II & 45^\circ & \text{Circular} & 0^\circ & \text{Linear} \\
III & 0^\circ & \text{Linear} & 45^\circ & \text{Circular} \\
IV & 90^\circ & \text{Linear} & 20^\circ & \text{Elliptic} \\
\end{array}
\]

For the elliptic polarization: 
\[
\frac{\text{Long axis}}{\text{Short axis}} = \frac{\sin 20^\circ}{\cos 20^\circ} = \tan 20^\circ = 0.36
\]

P6-15: If \( \delta = \frac{2\pi}{\lambda_0} \Delta n \cdot d \) then \( I = I_0 \sin^2 \left( \frac{\delta}{2} \right) \)

P6-16: a) 85 mm b) 2.0 mm c) linear at 45° and at 90°, \( I = 0 \)

P6-17: See FoP Figure 6.6-3, the angle between the beams is 11.4°

P6-18: 0.0096

P6-19: See FoP Figure 6.6-3, the angle between the beams is 5.8°