Light diffusion in turbid media

Medical Optics

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Steady State Diffusion

In steady state diffusion the time dependence is equal to zero and the equation for a point source is

$$\nabla^2 f(\mathbf{r}) - m_{eff}{}^2 f(\mathbf{r}) = S(\mathbf{r})$$

This yields a solution for a infinite homogenous medium as:

$$f(\mathbf{r}) = f(\mathbf{r}=\mathbf{0})\frac{1}{|\mathbf{r}|}\exp(-m_{eff}\cdot|\mathbf{r}|) = \frac{Pm_{eff}{}^2}{4pm_a}\frac{1}{|\mathbf{r}|}\exp(-m_{eff}\cdot|\mathbf{r}|)$$

Solution to steady state diffusion

$$f(\mathbf{r}) = \frac{P m_{eff}^{\ 2}}{4 p m_a} \frac{1}{|\mathbf{r}|} \exp(-m_{eff} \cdot |\mathbf{r}|)$$

where
$$\begin{cases} m_{eff} = \sqrt{3 m_a (m_a + m_s(1-g))} \\ m_s{}' = m_s(1-g) \\ D = \dfrac{1}{3(m_a + m_s(1-g))} = \dfrac{m_a}{m_{eff}^{\ 2}} \end{cases}$$

Steady-state solution

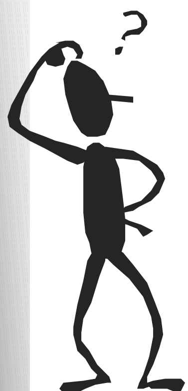
A simple check that this expression satisfies the steady-state diffusion equation follows:

$$\nabla^2 f(\mathbf{r}) = \frac{1}{r} \frac{\P^2}{\P r^2}\left(r f\right) = m_{eff}^{\ 2} f(\mathbf{r})$$

$$\Rightarrow$$

$$-\nabla^2 f(\mathbf{r}) + m_{eff}^{\ 2} f(\mathbf{r}) = d(0)$$

Example 1: PDT dosimetry



During PDT, assume that 2.15 J/cm^3 need to be absorbed to kill the tissue. Calculate the volume killed for a 15 minutes treatment from an isotropic interstitial fibre-tip emitting 160 mW light at 635 nm. Assume the optical properties to be (μ_a=1 mm^{-1}, μ_s'=100 mm^{-1})

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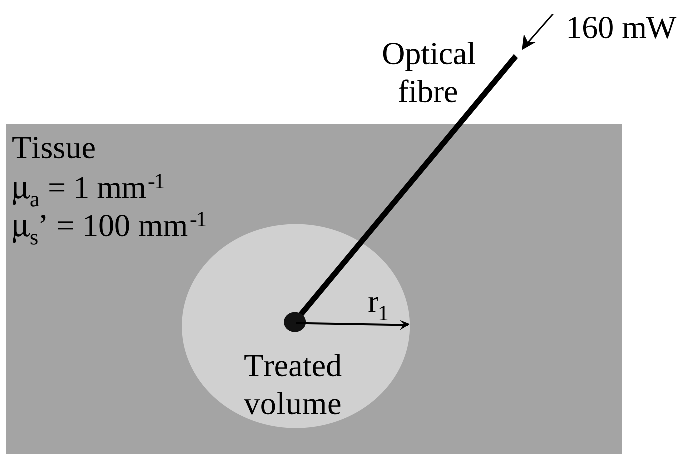
Interstitial PDT geometry



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Guidelines for solution of Ex 1

First, consider the solution to the steady-state diffusion equation as valid for this problem.

$$f(\mathbf{r}) = \frac{P m_{eff}^{2}}{4 p m_a} \frac{1}{|\mathbf{r}|} \exp(-m_{eff} \cdot |\mathbf{r}|)$$

with $m_{eff} = \sqrt{3 m_a (m_a + m_s')}$

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Guidelines cont'd (II)

Secondly, consider the absorbed power density $a(\mathbf{r})$ (mW/mm^3) to be: $a(\mathbf{r}) = m_a f(\mathbf{r})$

Thirdly, the absorbed energy density $A(\mathbf{r})$ depends on the treatment time T as $A(\mathbf{r}) = T \cdot m_a f(\mathbf{r})$

Insert all values in the equation:

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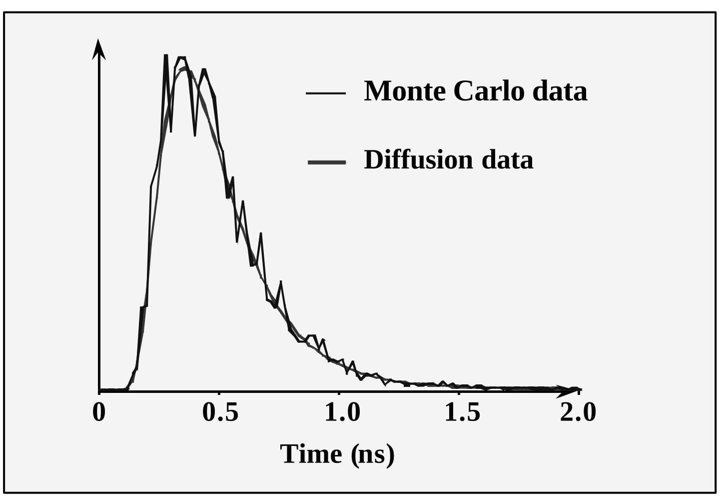
Guidelines cont'd (III)

This will result in an equation of the form:

$$\frac{1}{r} e^{-m_{eff} r} = C$$

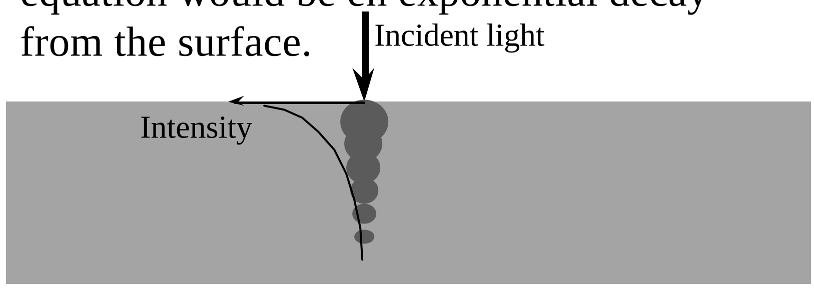
where C is a value calculated from all parameters given. It is difficult to find an expression for r from this equation, and it is easier to solve the problem graphically or iteratively.

Diffusion Equation



Source term in Diff. Eq.

- If we consider a pencil beam incident perpendicular on a semi-infinite volume of tissue, a useful source term in the diffusion equation would be en exponential decay from the surface.



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Source term for narrow beam incident light

To simplify the problem, we first assume that all the incident photons are initially scattered at a single depth of

$$z_0 = 1/m_s' = \left[(1-g)m_s\right]^{-1}$$

The reduced scattering coefficient μ_s’ can be regarded as an effective isotropic scattering coefficient that represents the cumulative effect of several forward-scattering events.

Thus, z_o corresponds to an isotropic source at the depth of one reduced scattering coefficient

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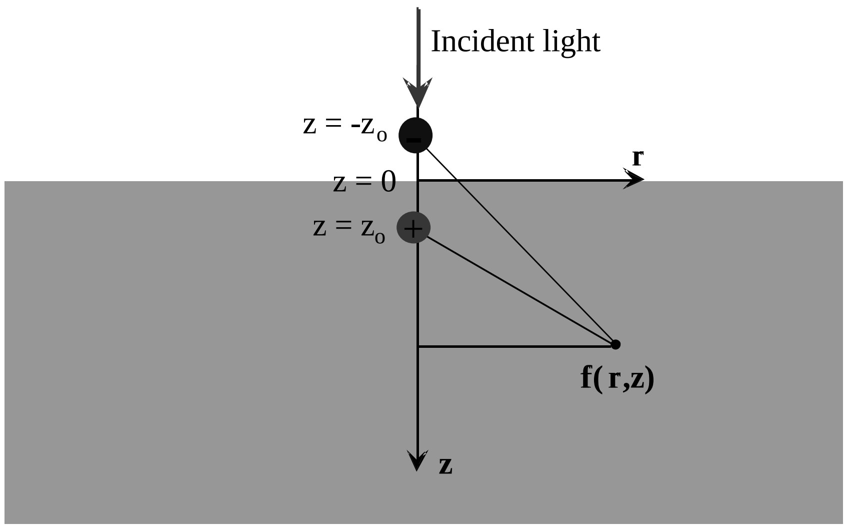
Boundary Condition

- In solving the diffusion equation for anything else than for an infinite homogenous medium, the boundary conditions must be met.
- A simple condition for the boundary that frequently provides accurate results is to assume that the fluence rate is zero at a boundary between the turbid and a non-scattering medium

Introduction of Dipole sources

- To treat the boundary conditions, mirror sources are introduced. They are positive and negative and positioned above and below the slab in a configuration giving zero fluence rate at the boundaries.
- In reality, there is usually a refractive index mismatch between the turbid medium (tissue) and the medium outside (air) which gives rise to Fresnel reflection at the boundary. A significant fraction of the radiant energy incident upon the boundary from inside will be reflected back. In this case it is unphysical to have zero radiance at the boundary.

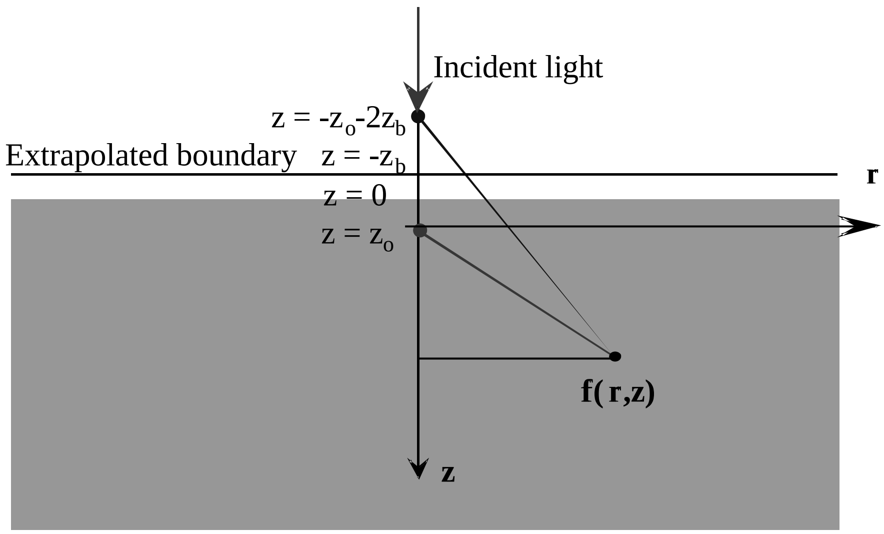
Point source geometry



Extrapolated Boundary Condition

- One model to correct for the Fresnel reflection is to use an so called Extrapolated-Boundary condition. In this approach the boundary, where the fluence rate is zero, is located some distance outside the tissue.
- This distance is a function of the effective Fresnel reflection coefficient and z_0. It varies from $0.7z_0$ for the same refractive indices of the two media to $2z_0$ for a tissue (n=1.4): air (n=1.0) boundary. This boundary condition model has been shown to yield good results.

Extrapolated Boundary



Use of Extrapolated Boundary Condition

- When the observation of the radiance is made at a distance from the source that is considerably larger than the extrapolated boundary length, the zero radiance at the physical boundary is a valid approximation.
- However, at shorter distances the use of an extrapolated boundary may be necessary to obtain accurate results.
- ***Note though,*** at too short distances the diffusion approximation is not valid anyway.

Fluence Rate in a Semi-infinite Homogenous Medium

The fluence rate in a semi-infinite medium can be written as a sum of two contributions, one from the real source and one from the artificial negative mirror source introduced to meet the boundary condition. Neglecting the extrapolated boundary effect, this gives:

$$f(r,z,t) =$$

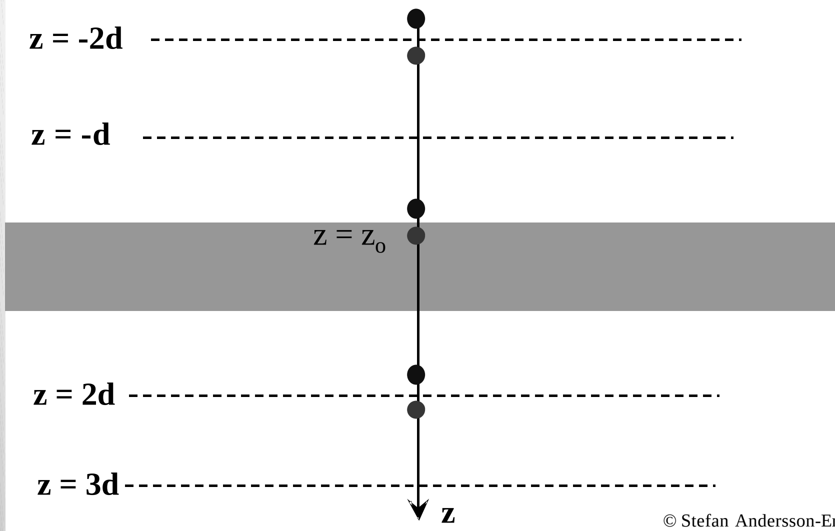
$$= c(4pDct)^{-3/2}\exp(-m_a ct)\left\{\exp\left[-\frac{(z-z_0)^2+r^2}{4Dct}\right]-\exp\left[-\frac{(z+z_0)^2+r^2}{4Dct}\right]\right\}$$

Boundary Conditions in the Slab Geometry

If we now consider a slab geometry instead of a semi-infinite medium, it is more complex to fulfil the boundary condition for both surfaces. The laser illumination gives rise to a point source located at z_0 along the z-axis into the slab of tissue. One now need to mirror the source in multiple planes. This means that the source term in the diffusion equation is now a sum of positive and negative sources located above and below the slab.

$$q_0 = vSd(t)\sum_{k=-\infty}^{+\infty}\left\{d(\mathbf{r}-[2kd+z_0]\mathbf{e_z})-d(\mathbf{r}-[2kd-z_0]\mathbf{e_z})\right\}$$

Mirror sources in the slab geometry



Diffuse Reflectance from a Semi-infinite Medium

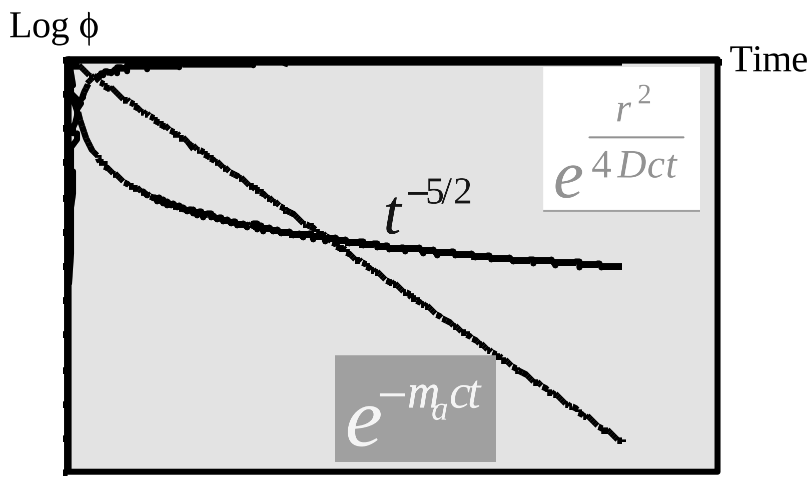
We had earlier that , $\phi(\mathbf{r},t) = ch\nu\rho(\mathbf{r},t)$ and

$\mathbf{J} = -c\,D\,\nabla\rho$ giving the power reaching the surface (in cylindrical co-ordinates):

$$R(r,t) = h\nu\mathbf{J}(r,0,t)\cdot\mathbf{n} = -D\frac{\partial}{\partial z}\phi(r,z,t)\Big|_{z=0} =$$

$$= (4\pi Dc)^{-3/2} z_0 t^{-5/2}\exp(-\mu_a ct)\exp(-\frac{r^2+z_0{}^2}{4Dct})$$

Time-dependent factors in the diffuse reflectance



Final slope of Diffuse Reflectance Curve

Under the assumption that $r^2 \rangle\rangle z_0{}^2$ the final slope for the diffuse reflectance can be calculated to be:

$$\frac{d}{dt}\ln R(r,t) = -\frac{5}{2t} - m_a c + \frac{r^2}{4Dct^2}$$

The final slope will thus be $\lim_{t\to 0}\frac{d}{dt}\ln R(r,t) = -m_a c$

Total Diffuse Reflectance from a Semi-infinite Homogenous Medium

To obtain the total diffuse reflectance from a semi-infinite homogenous medium one can integrate the function R(,t) over the surface:

$$R(t) = \int_0^{\infty} R(r,t)2p\,r\,dr =$$

$$= (4pDc)^{-1/2} z_0 t^{-3/2} \exp(-m_a ct) \exp(-\frac{z_0^{\ 2}}{4Dct})$$

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Diffuse Reflectance from a Homogenous Slab

In the same way as described previously following the derivation of the fluence rate, the diffuse reflectance from a slab illuminated with a short laser pulse at t=0 at (ρ,z)=(0,0) can be calculated to be:

$$R(r,d,t) = (4pDc)^{-3/2} t^{-5/2} \exp(-m_a ct) \exp\left(-\frac{r^2}{4Dct}\right) \times$$

$$\left\{ z_0 \exp\left[-\frac{z_0^{\ 2}}{4Dct}\right] - (2d - z_0)\exp\left[-\frac{(2d-z_0)^2}{4Dct}\right] + (2d+z_0)\exp\left[-\frac{(2d+z_0)^2}{4Dct}\right] \right\}$$

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Observations for the Diffuse Reflectance

- The loss of photons from the back surface causes the diffuse reflectance to decrease more rapidly with time.

- The total diffuse reflectance R(t) has a slightly different time-dependence than R(ρ,t)

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Total Diffuse Reflectance from a Homogenous Slab

By integrating R(ρ,d,t) over the surface area one gets the total diffuse reflectance:

$$R(d,t) = (4pDc)^{-1/2} t^{-3/2} \exp(-m_a ct) \times$$

$$\left\{ z_0 \exp\left(-\frac{z_0{}^2}{4Dct} \right) - (2d - z_0)\exp\left[-\frac{(2d-z_0)^2}{4Dct} \right] + (2d + z_0)\exp\left[-\frac{(2d+z_0)^2}{4Dct} \right] \right\}$$

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Diffuse Transmittance through a Homogenous Slab

In the same way as described for the diffuse reflectance, the diffuse transmittance through a slab of thickness d illuminated with a short laser pulse at t=0 at (ρ,z)=(0,0) can be calculated:

$$T(\rho,d,t) = (4\pi Dc)^{-3/2}\,t^{-5/2}\,\exp(-\mu_a ct)\exp(-\frac{\rho^2}{4Dct})\times$$

$$\left\{(d-z_0)\exp\left[-\frac{(d-z_0)^2}{4Dct}\right]-(d+z_0)\exp\left[-\frac{(d+z_0)^2}{4Dct}\right]+\right.$$

$$\left.+(3d-z_0)\exp\left[-\frac{(3d-z_0)^2}{4Dct}\right]-(3d+z_0)\exp\left[-\frac{(3d+z_0)^2}{4Dct}\right]\right\}$$

Total Diffuse Transmittance through a Homogenous Slab

By integrating T(ρ,d,t) over the surface area one gets the total diffuse transmittance:

$$T(d,t) = (4\pi Dc)^{-1/2}\,t^{-3/2}\,\exp(-\mu_a ct)\times$$

$$\left\{(d-z_0)\exp\left(-\frac{(d-z_0)^2}{4Dct}\right)-(d+z_0)\exp\left[-\frac{(d+z_0)^2}{4Dct}\right]+\right.$$

$$\left.+(3d-z_0)\exp\left[-\frac{(3d-z_0)^2}{4Dct}\right]-(3d+z_0)\exp\left[-\frac{(3d+z_0)^2}{4Dct}\right]\right\}$$

The Time-independent Diffusion Equation

The time-independent diffusion approximation can be written as:

$$-D\nabla^2 f(\mathbf{r}) + m_a f(\mathbf{r}) = S(\mathbf{r})$$

With the source in **r**=**0**, this equation has the solution

$$f(r) = \frac{S}{4pDr}\exp(m_{eff}\, r) \quad \text{, where} \quad m_{eff} = \sqrt{\frac{m_a}{D}}$$

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The Time-independent Fluence Rate

This solution would yield the following fluence inside a semi-infinite homogenous medium:

$$f(r, z) =$$

$$= \frac{1}{4pD}\left\{ \frac{\exp\left\{-m_{eff}\left[(z - z_0)^2 + r^2\right]^{1/2}\right\}}{\left[(z - z_0)^2 + r^2\right]^{1/2}} - \frac{\exp\left\{-m_{eff}\left[(z + z_0)^2 + r^2\right]^{1/2}\right\}}{\left[(z + z_0)^2 + r^2\right]^{1/2}} \right\}$$

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