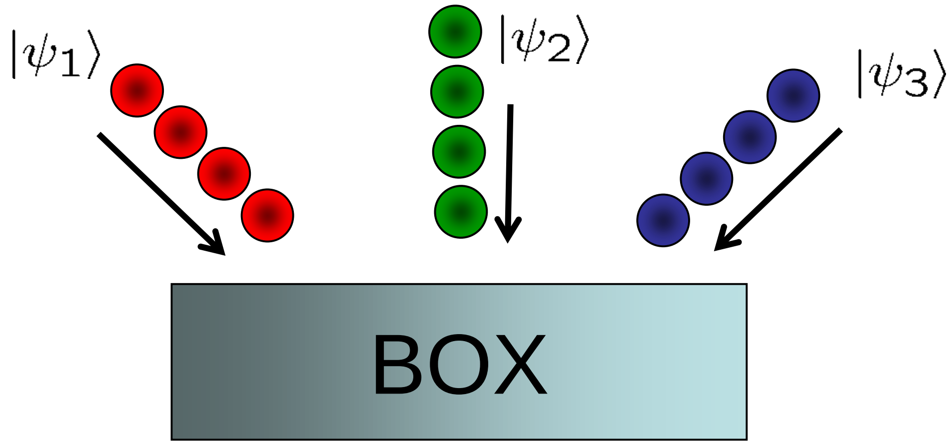
2.4 Density operator/matrix

Ensemble of *pure states* gives a *mixed state*



The *density operator* or *density matrix* ρ for the ensemble or mixture of normalized states $|\psi_i\rangle$ with probabilities p_i is given by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad \sum_i p_i = 1$$

Note: The set of states $|\psi_i\rangle$ and probabilites p_i are not unique:

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \frac{1}{2}\left[|a\rangle\langle a| + |b\rangle\langle b|\right] \qquad |a/b\rangle = \frac{1}{2}\left[\sqrt{3}|0\rangle \pm |1\rangle\right]$$

Composition

If we have systems numbered 1 through n, and system i is in state ρ_i , the state of the total system is

$$\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n$$

General properties

An operator ρ is a density operator if (and only if):

1) ρ has trace equal to one.
2) ρ is Hermitian
3) ρ is a positive operator.

Derivation: Show general properties from $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

Example: General single qubit density matrix, decomposition.

Derivation: Exercise 2.71, purity of a state ρ .

Time development

For an individual state $|\psi_i\rangle$, with U the unitary time evolution operator

$$|\psi_i\rangle \to U|\psi_i\rangle \qquad \Rightarrow$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \to \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger$$

Measurement

We perform a general measurement described by M_m. If the system is in state $|\psi_i\rangle$, the (conditional) probability to get m is

$$p(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = \mathrm{tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|)$$

The total probability to get m when measuring on ρ is then

$$\sum_i p_i p(m|i) = \sum_i p_i \mathrm{tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) = \mathrm{tr}(M_m^\dagger M_m \rho)$$

Post-measurement state

For an initial state $|\psi_i\rangle$, the state after measuring m is

$$|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}$$

The total state after measuring m on ρ is

$$\rho_m = \sum_i \frac{p_i p(m|i)|\psi_i^m\rangle\langle\psi_i^m|}{p(m)}$$

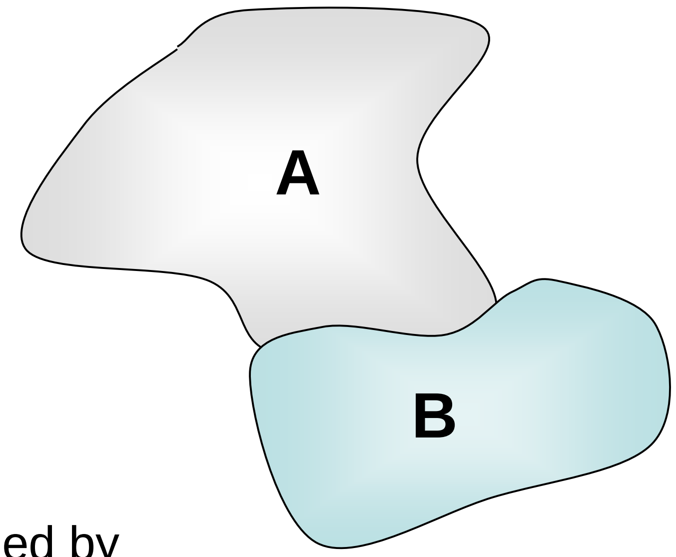
The normlization condition $\mathrm{tr}(\rho_m) = 1$ gives the denominator

$$p(m) = \sum_i p_i p(m|i)$$

Inserting known expressions, this gives

$$\rho_m = \sum_i p_i \frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)} = \frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}$$

Reduced density operator/matrix



The composite, total system AB is in the state ρ_{AB}.

The *reduced density operator* of system A is by definition

$$\rho_A = \mathrm{tr}_B\,(\rho_{AB})$$

where the *partial trace* over B is defined by

$$\mathrm{tr}_B\,(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2|\mathrm{tr}(|b_1\rangle\langle b_2|)$$

with $|a_1\rangle, |a_2\rangle \ \ (|b_1\rangle, |b_2\rangle)$ any vectors in A (B), and the linearity property of the trace.

The reduced density operator describes completely all the properties/outcomes of measurements of the system A, given that system B is left unobserved (”tracing out” system B)

Derivation: Properties of reduced density operator.

Derivation: Reduced density matrix for Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$

2.5 Schmidt decomposition and purification

Schmidt decomposition

For a pure state $|\psi\rangle$ in the composite system AB, there exists orthonormal bases $|i_A\rangle$ and $|i_B\rangle$ (*Schmidt bases*) for systems A and B, such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

with λ_i the real, non-negative Schmidt coefficients and

$$\sum_i \lambda_i^2 = 1$$

for $|\psi\rangle$ normalized, $\langle\psi|\psi\rangle = 1$.

Properties

The reduced density matrices ρ_A, ρ_B have the same eigenvalues

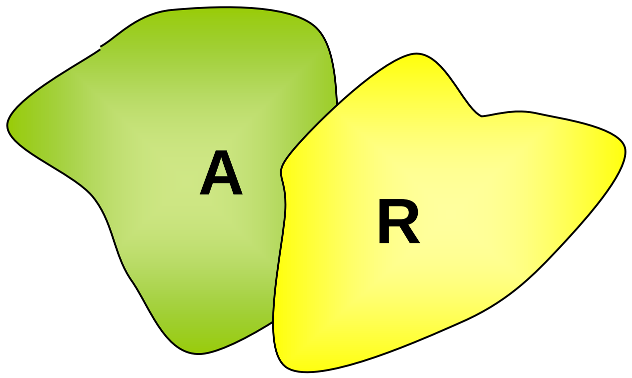
$$\rho_A = \mathrm{tr}_B\left(|\psi\rangle\langle\psi|\right) = \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \qquad \rho_B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

The *Schmidt number* is the number of non-zero Schmidt coefficients.

Purification

Consider a system A in state ρ_A.

It is possible to introduce an additional system R and to define a pure state $|AR\rangle$, such that the reduced density operator for A is



$$\mathrm{tr}_R(|AR\rangle\langle AR|) = \rho_A$$

This is called *purification*.

We can construct $|AR\rangle$ by first noting that we can spectrally decompose

$$\rho_A = \sum_i p_i |i_A\rangle\langle i_A|$$

By taking R to have the same dimensions as A we can define

$$|AR\rangle = \sum_i \sqrt{p_i}|i_A\rangle|i_R\rangle$$

This then gives

$$\mathrm{tr}_R(|AR\rangle\langle AR|) = \sum_{ij}\sqrt{p_i p_j}|i_A\rangle\langle j_A|\mathrm{tr}(|i_R\rangle\langle j_R|) = \sum_i p_i|i_A\rangle\langle i_A| = \rho_A$$